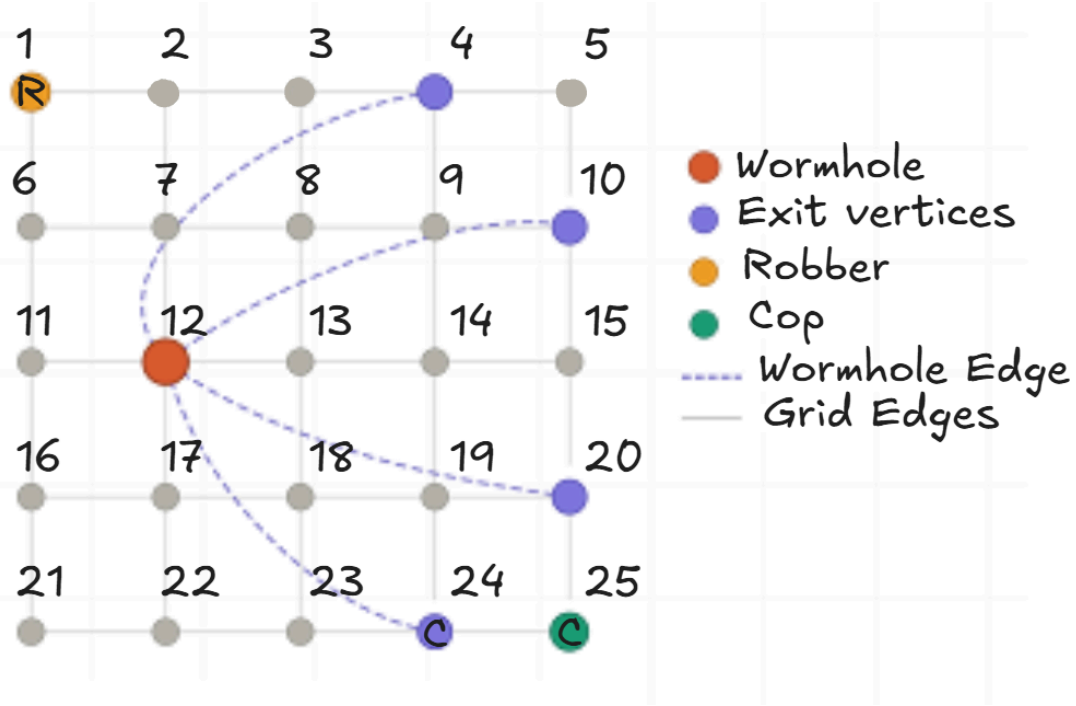
Wormhole Cops and Robbers on Grid Graphs

Determining Optimal Cop Numbers by Distance Threshold



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Math Research · Period 7/8

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I. Research Question

Given n x n grid graph whose vertices are labeled $1, 2, 3, \ldots, n^2$. A game of Cops and Robbers is played on this graph with two cops and one robber. Initially the robber starts at vertex 1, while the two cops start on vertices n^2 and $n^2 - 1$. The robber has to choose one wormhole vertex where it connects to all vertices sharing a factor greater than 1 and greater than k, such that k is an integer greater than 1. On each turn a player may move to 1 edge at time, starting with the robber. For which values of k does capturing the robber become impossible and increase the amount of cops to three?

II. Intro

Imagine a game of chess where your opponent can move their King to any square on the board once every move. On a standard graph, a set of cops attempt to capture a single robber by moving along edges from vertices to vertices. On graphs, such as grid graphs, two cops are typically sufficient to guarantee capture, no matter what the robber strategy is. However, this is way too easy for the cops, so what if we introduce a single “wormhole” vertex, a teleportation vertex connecting vertices of the graph with requirements of a minimum manhattan distance k and sharing common factors, and the pursuit changes from a simple, grid chase into a complex chaotic game. This question explores concepts of graph theory, and combinatorics to analyze the most optimal decisions in the game and explores the “Wormhole Cops and Robber” game, aiming to find the exact value of k, distance required between two vertices to teleport, where two cops are no longer adequate, introducing a third cop.

III. History

The game of Cops and Robbers is a well-studied game in graph theory. This was first formally introduced in work by G. Hahn and R. Nowakowski. In this game, a set of cops and a single robber occupy vertices of a graph and move alternately along edges. The goal of the cop is to land on the same vertex as the robber, aka capturing them, while the robber attempts to evade capture indefinitely. The core concept of this game is the distance k on a n by n graph, which is the minimum number of cops required to guarantee capture of the robber regardless of what strategy they intend to use. Finding the number of cops in different types of graph has been the major topic in graph theory and combinatorics. Further research, including work by Anthony Bonato and Richard J. Nowakowski has explored the complexity of determining the number of cops, showing that the problem becomes increasingly difficult to generalize for any graphs. These investigations demonstrate how small changes in a graph can significantly affect strategies and outcomes.

IV. Mathematical Background:

Graphs

A graph $G = (V, E)$ consists of a set of vertices V and a set of edges E, where each edge connects two vertices. Graphs are used in mathematics and computer science to model relationships. This investigation uses a specific type of graph called a grid graph.

A grid graph is an n x n grid graph $G_n = (V, E)$ has vertex set $V = \{1, 2, 3, \ldots, n^2\}$. Two vertices are adjacent if and only if they are horizontally or vertically neighbors on the grid.

Manhattan Distance

The Manhattan Distance measures the shortest path between two vertices moving only along grid edges. For two vertices $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ in a n x n grid, the Manhattan distance is

$$d(v_1, v_2) = |x_2 - x_1| + |y_2 - y_1|.$$

The diameter of a graph is the maximum distance between any two vertices. For an n x n grid, the diameter is computed through two opposite corners as vertex 1 at (1,1) and vertex (n,n).

$$diam(G_n) = |(1) - (n)| + |(1) - (n)| = 2(n - 1)$$

For example $diam(G_4) = 2(3) = 6$ and $diam(G_5) = 2(4) = 8$

GCD

The greatest common divisor of two integers is the largest integer that divides both numbers without a remainder. For two positive integers a and b, gcd(a,b) is the largest positive integer d such that $d \mid a$ and $d \mid b$. Two integers are called coprime if $gcd(a, b) = 1$.

The GCD can be efficiently computed using the Euclidean Algorithm. An algorithm where you repeatedly replace the larger number with the remainder when divided by the smaller number, until the remainder is zero.

$gcd(12, 8)$ using Euclidean Algorithm:

$$12 = 1 * 8 + 4 \Leftrightarrow gcd(12, 8) = gcd(8, 4)$$

$$8 = 2 * 4 + 0 \Leftrightarrow gcd(8, 4) = 4$$

Therefore $gcd(12, 8) = 4$. Since gcd > 1, vertices 12 and 8 share a common factor.

Wormhole Edges

This investigation focuses on a grid graph with additional wormhole edges. These edges connect vertex pairs that share a common factor, but are far apart in the grid. The threshold distance is controlled by a parameter k, an integer greater than 1.

A wormhole exists between v_1 and v_2 if G_n if and only if

(1) $gcd(v_1, v_2) > 1$

(2) $d(v_1, v_2) \geq k$

The robber in the Cops and Robbers may choose one vertex to serve as the wormhole vertex w. From w, the robber can teleport to any vertex reachable by a wormhole edge. The collection of these reachable vertices is called the exit set, $E_k(w)$.

For a chosen wormhole vertex and threshold k, the exit set is

$$E_k(w) = \{ v \in V: \ gcd(w, \ v) \ > \ 1 \ and \ d(w, \ v) \ \geq \ k\}$$

Example:

The exit set for wormhole vertex w=6 in G_4 with k=3. Since $6 = 2 * 3$, vertex 6 shares a factor with every multiple of 2 or 3. This table below shows the candidates

Table 1: Exit set computation for w = 6, k = 3 in G_4

Vertex	Position	gcd(6, v)	Distance from v6	In exit set (k=3)
4	(1,4)	2	3	Yes
12	(3,4)	6	3	Yes
8	(2,4)	2	4	Yes
15	(4,3)	3	3	Yes
16	(4,4)	2	4	Yes
2	(1,2)	2	1	No
9	(3,1)	3	2	No
10	(3,2)	2	1	No
14	(4,2)	2	2	No

3	(1,3)	3	2	No

V. Investigation

Case Study: n = 4

Case k = 2

When k = 2, a wormhole at some vertex w reaches every vertex with gcd(w,v) > 1 and d(w,v) > 2. The even-vertex network is strong: any two even vertices that aren't close neighbors are connected.

Let's consider the wormhole placed at vertex 6 at position (2,2). Vertex 6 = 2 × 3, so it shares a factor with every multiple of 2 or 3. The multiples of 2 or 3 that are in the set of vertices are: 2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16.

Table 2: Exit set computation for w = 6, k = 2 in G_4

Vertex	Position	Distance from vertex 6
2	(1,2)	1
3	(1,3)	2
4	(1,4)	3
8	(2,4)	2
9	(3,1)	2
10	(3,2)	1
12	(3,4)	3
14	(4,2)	2
15	(4,3)	3
16	(4,4)	4

Removing those within distance of 2 leaves vertices: {4, 12,15,16}

The key observation I found is that

$$d(4, 15) = |1 - 4| + |4 - 3| = 4$$
$$d(12, 16) = |3 - 4| + |4 - 4| = 1$$
$$d(15, 16) = 1$$

The diameter of this exit set is 2, meaning a cop at vertex 16 is within distance 1 of all exits.

However, the wormhole degree is large enough that the robber has escape options scattered around the grid from the cops' starting position at 15 and 16. The robber, starting at vertex 1, can move toward vertex 2 in 2 moves. Meanwhile, the cops begin at 15 and 16. Before a cop reaches vertex 6 or covers all exits, the robber teleports to the exit farthest from both cops. The robber strategy at k=2 is to move to vertex 6, then teleport to vertex 12 which is 3 moves from cop 1 at vertex 12, but both cops would need to be adjacent to vertex 6 or block all exits. Since they cannot split their coverage zones, the robber can get away.

Case k=3:

When k = 3, wormhole edges require d > 3, so the robber can only teleport to vertices that are at least 4 steps away. On a 4×4 with diameter 6, the exits become narrower.

If we consider vertex 6 at (2,2): when k = 3, we remove vertices

2 (d=1), **3** (d=2), **4** (d=3, excluded since d = k not > k), **8** (d=2), **9** (d=2), **10** (d=1), **12** (d=3, excluded), **14** (d=2), **15** (d=3, excluded).

This leaves only vertex 16.

With the wormhole degree reduced to 1, the robber's escape options are more narrow. The two cops can attempt to have one cop to cover the center of the board, while the other cop chases the robber. The cop that covers the center of the board can patrol the single vertex on a 4x4 grid, so two cops are sufficient for a 4x4 grid when k=3.

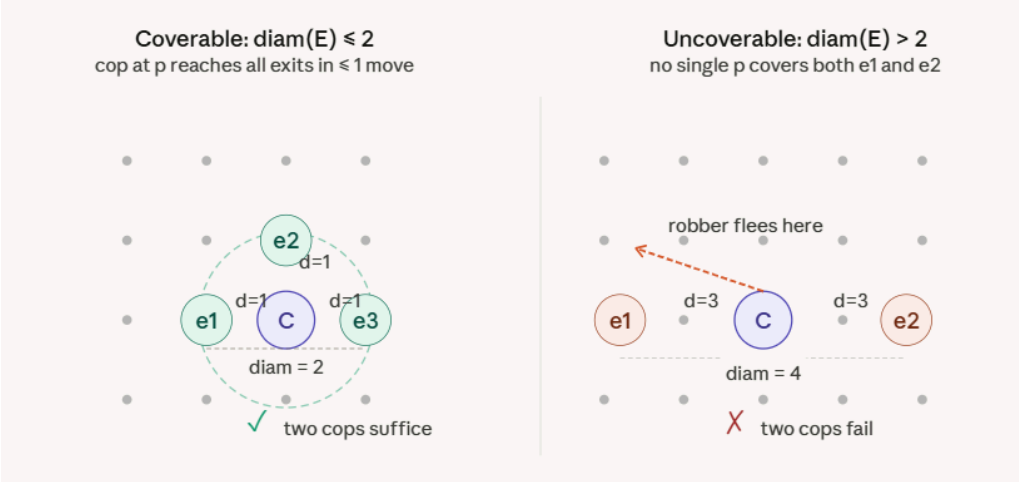
An additional case worth examining is wormhole vertex 10, since $10 = 2 \times 5$ connects to multiples of 2 and 5: {2, 4, 5, 6, 8, 12, 14, 15, 16}. When k = 3, removing those within distance ≤ 3 leaves only vertex 16. Two cops are also able to cover this with the same method.

Therefore for n = 4, k = 3 is needed to capture the robber for all wormhole placements.

Sufficiency Argument

We can make an argument that two cops suffice if one cop can occupy or be adjacent to the wormhole, and the second cop is positioned such that every exit vertex is within d=1 of that cop. Two cops fail if there exists two exit vertices: e_1 and e_2 such that $d(e_1, e_2) \geq 3$ because then no single cop position can be within distance of 1 for both simultaneously. In this case, one cop cannot cover all exits, so the robber can always choose to teleport to the farthest from the covering cop, guaranteeing escape.

- A set of exit vertices is convertible by one cop if there exists a vertex p on the grid such that $d(p, e) \leq 1$ for every value of e such that e must be in the set of exit vertices.
- If the diameter of the exit set is less than or equal to 2, then there exists a vertex p such that every exit vertex is within distance 1 of p, so one cop can control all exits.



Proof:

Let G be an n x n grid. Fixed wormhole vertex w, and let E(w) be the set of the exit vertices. If there exists a vertex p such that $d(p, e) \leq 1$ for all e in set E(w), then two cops can guarantee capture.

Let the first cop, C_1, move toward the wormhole vertex w. Once C_1 is at w or adjacent to it, the robber cannot safely use the wormhole. If a robber moves onto w, they are immediately captured or captured on the next move, so the wormhole is blocked.

Let C_2 move to vertex p, where every exit vertex is either p itself or adjacent to p. So if the robber teleports to any exit e, then C_2 is at distance less than or equal to 1.

Suppose the robber uses the wormhole and lands at some e. Since d(p, e) ≤ 1, either C_2 is already on e or C_2 moves to e on the next turn and captures the robber. Thus any wormhole exit results in inevitable capture.

With no wormhole, the robber is confined to standard grid movement. We know that a grid graph has 2 cops, so two cops can capture the robber by pursuing them. Therefore, the robber cannot escape.

Case Study: n = 5

The 5x5 grid has 25 vertices with the layout:

1(1,1) 2(1,2) 3(1,3) 4(1,4) 5(1,5)
6(2,1) 7(2,2) 8(2,3) 9(2,4) 10(2,5)
11(3,1) 12(3,2) 13(3,3) 14(3,4) 15(3,5)
16(4,1) 17(4,2) 18(4,3) 19(4,4) 20(4,5)
21(5,1) 22(5,2) 23(5,3) 24(5,4) 25(5,5)

The diameter of G(5) is 2(5-1)=8. The cops begin at vertices 24 and 25

Case k=3

Vertex 12 at position (3,2) is chosen as the wormhole because $12 = 2^2 \times 3$, meaning it shares a common factor with every multiple of 2 or 3. On the 5×5 grid this accounts for a large amount of all vertices, maximizing the potential exit set size. Additionally, its position between the center and edge of the grid allows exits to spread toward opposite corners, maximizing exit set diameter.

The multiples of 2 or 3 in the grid are {2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24}. We keep those at distance strictly greater than 3 from (3,2):

- Vertex 4 (1,4): $d = |1-3| + |4-2| = 4$
- Vertex 10 (2,5): $d = |2-3| + |5-2| = 4$
- Vertex 20 (4,5): $d = |4-3| + |5-2| = 4$
- Vertex 24 (5,4): $d = |5-3| + |4-2| = 4$

All other multiples of 2 or 3 are within distance ≤ 3 of (3,2) and are excluded. The exit set is {4, 10, 20, 24}.

We need to check whether the cops can cover all exits, so we compute the diameter of the exit set. The key pair is $d(4,24) = |1-5| + |4-4| = 4 > 2$. Since no single grid vertex can be within distance 1 of both vertex 4 and vertex 24 simultaneously, one cop cannot cover all exits. The robber's strategy is to travel to vertex 12 and teleport to whichever exit is farthest from the second cop. Therefore, two cops are insufficient at $k = 3$.

Case k=4

At $k = 4$, the wormhole edges require a distance greater than 4. Recomputing the exits from vertex 12 leaves only vertex 24 ($d = 4$, excluded since $d = k$ and not greater than k), so the exit set from vertex 12 is empty. Instead, consider a wormhole at vertex 6 at position (2,1). We check all multiples of 2 or 3 in the grid for distance greater than 4 from (2,1):

- Vertex 15 (3,5): $d = |3-2| + |5-1| = 5$
- Vertex 20 (4,5): $d = |4-2| + |5-1| = 6$
- Vertex 24 (5,4): $d = |5-2| + |4-1| = 6$

The exit set is {15, 20, 24}. The diameter of this exit set is $d(15,24) = |3-5| + |5-4| = 3 > 2$. No single cop position covers all exits within distance 1, so the robber can always teleport to an uncovered exit. Therefore, two cops are insufficient at $k = 4$.

Case k=5

We need to check which wormhole has the highest degree and most spread out exits. At $k = 5$, with grid diameter of 8, only vertex pairs near opposite corners have a distance greater than 5.

The strongest wormhole at $k = 5$ is vertex 2 at (1,2). This gives all the even vertices, and checking those with $d > 5$:

Vertex 24 (5,4) at $d = 4+2 = 6$
Vertex 20 (4,5) at $d = 3+3 = 6$

Vertex 22 (5,2) at d = 4+0 = 4 (excluded).

So the only exits are {20, 24}.

With only 2 exits, one cop sits on or adjacent to the wormhole and the second cop can cover both exits. Since d(20,24) = |4−5| + |5−4| = 2, the second cop positioned at (5,4) = vertex 24 is already adjacent to both exits. After any teleport, the second cop is at most 1 move away from capturing the robber. Therefore, two cops are sufficient at k = 5.

Case Study: n = 6

We can do this one more time to find the pattern between n and k. The 6x6 grid has 36 vertices and a diameter of 10.

1 2 3 4 5 6
7 8 9 10 11 12
13 14 15 16 17 18
19 20 21 22 23 24
25 26 27 28 29 30
31 32 33 34 35 36

Case k=4

The robber's optimal wormhole is vertex 12 at position (2,6), connecting to a large set of exit vertices. Vertex 12 = $2^2 \times 3$ connects to all multiples of 2 or 3, and its corner-adjacent position pushes exits toward the opposite side of the grid. Let's find the vertices that satisfy gcd(12, v) > 1 and d > 4 from (2,6):

Table 3: Exit set computation for w = 12, k = 4 in G_6

Vertex	Position	gcd(12,v)	Distance	In exit set
2	(1,2)	2	5	Yes
14	(3,2)	2	5	Yes
20	(4,2)	4	6	Yes
21	(4,3)	3	5	Yes
26	(5,2)	2	7	Yes

Vertex	Position	gcd(12,v)	Distance	In exit set
27	(5,3)	3	6	Yes
28	(5,4)	4	5	Yes
32	(6,2)	4	8	Yes
33	(6,3)	3	7	Yes
34	(6,4)	2	6	Yes
22	(4,4)	2	4	No (d = k)

Exit set: {2, 14, 20, 21, 26, 27, 28, 32, 33, 34}. Since d(2,33) = |1−6| + |2−3| = 6 > 2, the exit set diameter exceeds 2 and two cops are insufficient at k = 4.

Case k = 5

Using the same wormhole at vertex 12 at (2,6), we now keep only vertices with d > 5:

Table 4: Exit set computation for w = 12, k = 5 in G_6

Vertex	Position	Distance from (2,6)	In exit set
19	(4,1)	2+5 = 7	Yes
20	(4,2)	2+4 = 6	Yes
25	(5,1)	3+5 = 8	Yes
26	(5,2)	3+4 = 7	Yes
27	(5,3)	3+3 = 6	Yes
31	(6,1)	4+5 = 9	Yes
32	(6,2)	4+4 = 8	Yes
33	(6,3)	4+3 = 7	Yes
34	(6,4)	4+2 = 6	Yes

Vertex	Position	Distance from (2,6)	In exit set
21	(4,3)	2+3 = 5	No (d = k)
28	(5,4)	3+2 = 5	No (d = k)

Exit set: {19, 20, 25, 26, 27, 31, 32, 33, 34}. Since d(19,34) = |4−6| + |1−4| = 5 > 2, two cops are insufficient at k = 5.

Case k = 6

Using the same wormhole at vertex 12 at (2,6), we now keep only vertices with d > 6:

Table 5: Exit set computation for w = 12, k = 6 in G_6

Vertex	Position	Distance from (2,6)	In exit set
19	(4,1)	7	Yes
25	(5,1)	8	Yes
26	(5,2)	7	Yes
31	(6,1)	9	Yes
32	(6,2)	8	Yes
33	(6,3)	7	Yes
20	(4,2)	6	No (d = k)
27	(5,3)	6	No (d = k)
34	(6,4)	6	No (d = k)

Exit set: {19, 25, 26, 31, 32, 33}. Since d(19,33) = |4−6| + |1−3| = 4 > 2, two cops are insufficient at k = 6.

Case k = 7

The best wormhole is at vertex 2 at (1,2):

Table 6: Exit set computation for w = 2, k = 7 in G_6

Vertex	**Position**	**Distance from vertex 2**
30	(5,6)	4+4 = 8
36	(6,6)	5+4 = 9
24	(4,6)	3+4 = 7
34	(6,4)	5+2 = 7

The exit set is {30, 36} and d(30,36) = |5−6| + |6−6| = 1 ≤ 2. A cop at vertex 36 is within distance 1 of both exits. Two cops suffice at k = 7.

Generalization:

Table 7: k required to capture n x n grid

n	k required to capture the robber
4	3
5	5
6	7

At first, it seemed that we needed k = n in order to capture the robber because when n = 5, k = 5. But, this wasn't the case for n = 4 or n = 6 which didn't follow the same pattern. Another thought that came to mind was the grid diameter/2 + 1 because the diameter for n = 5 was 8 and for n = 6 it was 10. 8/2 + 1 = 5 and 10/2 + 1 = 6. This was not the case for n = 4, because the diameter was 6 and 6/2 + 1 = 4.

Conjecture: For an n × n grid with n ≥ 4, two cops are sufficient to capture the robber if $k \geq 2n - 5$.

2(4)−5 = 3
2(5)−5 = 5
2(6)−5 = 7
2(7)−5 = 9

$$2(8)-5 = 11$$

We wanted to find whether two cops succeeded in capturing the robber due to the diameter of the exit set, the maximum distance between any two exits from the robber's chosen wormhole. When this diameter is less than or equal to 2, a single cop can position themselves to cover the exit vertices within a distance of 1. When the diameter exceeds 2, the robber can always teleport to whichever exit is farthest from the covering cop.

For a wormhole placed near one corner of the grid, the exits are pushed to the opposite corner. As k increases, these exits get compressed into smaller regions. Since the wormhole condition is strict (d > k), all exits must lie at distance at least k + 1 from the wormhole. The maximum possible exit diameter from any wormhole is therefore:

$$diam(G) - (k + 1) = 2(n - 1) - k - 1 = 2n - k - 3$$

Setting this bound at most 2:

$$2n - k - 3 \leq 2$$
$$k \geq 2n - 5$$

This matches the observed threshold exactly. Note that the strict inequality d > k in the wormhole definition is what shifts the bound by 1 compared to the naive formula — vertices at exactly distance k are not exits and do not contribute to the exit spread.

At k = 2n − 5, this gives 2n − (2n−5) − 3 = 2, meaning exits can be spread exactly distance 2 apart, which is coverable. At k = 2n − 6, it gives 2n − (2n−6) − 3 = 3, which is not coverable by one cop alone.

On G_7, the formula predicts the threshold at k = 9.

At k = 8 (two cops should fail): Wormhole at vertex 6 at (1,6). Since 6 = 2 × 3, we check multiples of 2 or 3 with distance greater than 8 from (1,6):

- Vertex 36 (6,1): d = |6−1| + |1−6| = 10
- Vertex 43 (7,1): d = |7−1| + |1−6| = 11
- Vertex 44 (7,2): d = |7−1| + |2−6| = 10
- Vertex 45 (7,3): d = |7−1| + |3−6| = 9

Exit set: {36, 43, 44, 45}. The diameter d(36,45) = |6−7| + |1−3| = 3 > 2. Two cops fail.

At k = 9 (two cops should succeed): Same wormhole. Now vertex 45 has d = 9, which is excluded since d = k not > k. The exit set becomes {36, 43, 44} and d(36,44) = |6−7| + |1−2| = 2 ≤ 2. Two cops succeed.

On G_8, the formula predicts the threshold at k = 11.

At k = 10 (two cops should fail): Wormhole at vertex 2 at (1,2) on an 8×8 grid. Even vertices with distance greater than 10 from (1,2):

- Vertex 48 (6,8): d = 5+6 = 11
- Vertex 56 (7,8): d = 6+6 = 12
- Vertex 62 (8,6): d = 7+4 = 11
- Vertex 64 (8,8): d = 7+6 = 13

Exit set: {48, 56, 62, 64}. The diameter d(48,62) = |6−8| + |8−6| = 4 > 2. Two cops fail.

At k = 11 (two cops should succeed): Same wormhole. Now vertices 48 (d=11) and 62 (d=11) are excluded. Exit set: {56, 64}. The diameter d(56,64) = |7−8| + |8−8| = 1 ≤ 2. Two cops succeed.

Three cop questions.

We have found that two cops succeed if $k \geq 2n - 5$, meaning two cops fail for all $k < 2n - 5$. The original research question asks whether adding a third cop can solve those cases where 2 cops fail.

With three cops, one still guards the wormhole while the other two share exit coverage. Two cops can cover an exit set if it can be split into regions where each region has a diameter less than or equal to 2. Then, we can say that the total exit diameter is less than or equal to 4 with 3 cops:

$$2n - k - 3 \leq 4$$
$$k \geq 2n - 7$$

Using n = 5, k = 4: Exit set {15, 20, 24}. C_1 guards vertex 6. C_2 covers {15, 20} since d(15,20) = 1. C_3 covers vertex 24. Three cops suffice.

Using n = 5, k = 3: Exit set from vertex 12 is {4, 10, 20, 24}. C_1 guards vertex 12. C_2 covers {4, 10} since d(4,10) = 2. C_3 covers {20, 24}: d(20,24) = 2, sits at vertex 19 at (4,4). Three cops suffice.

However, there are limitations when three cops fail. For small k, the exit diameter is too large for 2 covering cops to divide into coverable regions.

At n = 6, k = 3: a wormhole at vertex 2 at (1,2) produces an exit set of eight vertices spread across the bottom right of the grid. No matter how two covering cops split these exits, at least one region has diameter greater than 2. Three cops are insufficient here.

C_1 must remain at or adjacent to w throughout the game. If no cop guards the wormhole, the robber may freely re-enter w and teleport repeatedly, preventing capture indefinitely.

The robber can only teleport to places that are far away. In an n × n grid, the maximum spread of exits is 2n − k − 3. In a grid, a cop catches a robber if they are 1 step away, meaning one cop can cover an exit region of diameter at most 2. To cover the whole spread using the cops who aren't guarding the wormhole, the number of covering cops needed is $\lceil (2n - k - 3) / 2 \rceil$, giving the total cop count as $1 + \lceil (2n - k - 3) / 2 \rceil$.

Condition	**Total cops needed**
$k \geq 2n - 5$	2
$k = 2n - 6$ or $2n - 7$	3
$k = 2n - 8$ or $2n - 9$	4
$k = 2n - 10$ or $2n - 11$	5

VI. Conclusion:

This investigation analyzed the Wormhole Cops and Robbers game on $n * n$ grid graph, seeking to find the threshold value k, the minimum wormhole distance, at which two can no longer guarantee capture of the robber, and whether a third cop can suffice capture.

The tool that was developed in this paper is the exit set diameter argument: two cops succeed against a wormhole at vertex w if and only if the exit set $E_k(w)$ has diameter at most 2, because a single can can then position within distance 1 of every exit. When the diameter exceeds 2, the

robber can always teleport to whichever exit is farthest from the covering cop, guaranteeing escape.

The maximum possible exit set diameter from any wormhole is bounded by $2(n-1)-k$, since all exits must lie at a distance greater than k from the wormhole, which compresses the exit vertices into a region of the grid whose diameter shrinks as k grows. Setting this bound at most 2 gives the threshold condition $k \geq 2n-4$.

The conjecture states that the maximum exit set diameter across all possible wormhole placements is exactly $2(n-1)-k$, and the exit is always coverable. This was verified only computationally for small n, but a general proof still remains.

Three-Cop Results

When $k = 2n-6$, the exit set diameter reaches 3. One cop guards the wormhole, the two remaining cops split the exit set into two regions of diameter at most 2, each covered by one cop. Three cops suffice at $k = 2n-6$. To be more general, the number of covering cops needed is $\lceil (2(n-1)-k)/2 \rceil$.

Answer to Research Question

Two cops fail to guarantee capture when $k < 2n-5$. Adding a third cop resolves the case $k = 2n-6$, but doesn't resolve all failing cases. For smaller k, the exit set is too large for any fixed number of covering cops to control, and the required number of cops grows with the exit set diameter.

VII. Extensions

Extending to non grid graph:

- Rectangular grid: ($m * n$ with $m \neq n$): the diameter would be $(m - 1) + (n - 1)$, and using the same methods, we could find a formula to generalize.
- 3D grid graphs: multi floor buildings or adding layers to graphs.

Multiple Wormholes: This paper allowed the robber to make one wormhole vertex. Instead introducing another wormhole vertex with each has its own exit vertex. The robber could then choose which wormhole to use, making their escape chances increased. The question becomes: for a fixed k and m wormholes, how many cops are needed?

Wormhole Placement by the Cops: An variant could allow one cop to choose the wormhole vertex instead of the robber. This switch allows fewer cops to succeed at smaller values of k. Prime Vertex Wormhole: This paper required $gcd(v, w) > 1$, connecting vertices that share any common factor. A more interesting variant would be requiring $gcd(w, v)$ to be prime. The distribution of primes among grid vertices gives more randomness and bigger advantage to the robber.

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Applications

Network Security and Pursuit Evasion

This can be applied for network security because The cops and Robbers game can be applied to pursuit evasion problems in network security, where an intruder (robber) moves through a network and defenders (cops) attempt to intercept them. The wormhole mechanic models a scenario where the intruder has access to encrypted tunnels (or VPN) between non-adjacent nodes. The threshold k corresponds to a minimum jump distance that makes a tunnel worth using. The cop number formula derived here suggests how many monitoring agents are needed depending on how powerful the intruder's tunneling routes are.

Robot Navigation and Search problems

In robotics, multi-robot search problems ask how many robots are needed to guarantee finding a moving target in a known environment. Grid graphs are common models for warehouse or indoor environments. A wormhole could model an elevator or conveyor belt, and using this paper can give us a starting point for how many pursuit robots are needed when the target has access to shortcuts.

VIII. References List

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