Math Research Writing
Mini-Research Paper

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I Statement of the Research Question

Given the integers $1, 2, 3, \ldots, 2005$, what is the sum of the distinct arrangements when the first term is 1, the last term is 2005, and the difference between any two consecutive terms is either 2 or 3?

II Origin of the Question

This question was from The Mandelbrot Competition October 2005 Round one Question 5. The question came from problems with permutations of numbers with limited common differences between consecutive terms. The questions we did in class on Knights tour of Hamiltonian path also was used to formulate a more complex form of the question. I was curious about what happens when you change the allowed differences, which led me to investigate the problem more beyond just the original question.

III Investigation of the Question

III.1 Graph

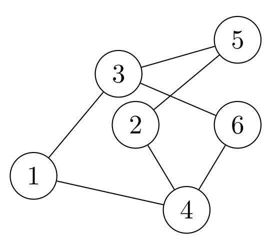
I began by modeling this as a graph theory problem. Each integer from 1 to n represents a vertex, and edges connect two integers if their difference is exactly 2 or 3. A valid arrangement corresponds to a path from vertex 1 to vertex n where we visit each vertex at most once.

Formally, the graph is defined as:

$$V = \{1, 2, 3, \ldots, n\}, \quad E = \{(a, b) \mid |a - b| = 2 \text{ or } |a - b| = 3\}$$

However, I quickly realized that trying to find all paths for $n = 2005$ directly would be tedious, so I decided to start with smaller values.

Modeling $n = 6$:



So the all valid arrangements are: $1 \to 3 \to 6$ and $1 \to 4 \to 6$

III.2 Table

I tested small values to find patterns:

n	$f(n)$	Sample Arrangements
1	1	1
2	0	
3	1	$1 \to 3$
4	1	$1 \to 4$
5	1	$1 \to 3 \to 5$
6	2	$1 \to 3 \to 6$, $1 \to 4 \to 6$
7	2	$1 \to 3 \to 5 \to 7$, $1 \to 4 \to 7$
8	3	$1 \to 3 \to 5 \to 8$, $1 \to 3 \to 6 \to 8$, $1 \to 4 \to 6 \to 8$

The case $n = 2$ stood out because there's no way to reach 2 from 1 using jumps of size 2 or 3. From 1, I can only reach 3 or 4, so 2 is unreachable.

III.3 Hypothesis of Recurrence Formula

My first thought was to use a computer program to run through all the values and find a pattern by starting at vertex 1 then jump 2 or 3 and recursively explore all paths. Finally, count the paths that succeeded in reach n

This worked well for small values of n, but when I tried computing $n = 30$ or $n = 50$, it took way too long to compute.

Now, backtracking to the original question I analyzed the sequence: $1, 0, 1, 1, 1, 2, 2, 3, \ldots$, my first guess was to check if it followed a Fibonacci pattern like $f(n) = f(n-1) + f(n-2)$.

Testing this hypothesis:

$$f(7) \stackrel{?}{=} f(6) + f(5) = 2 + 1 = 3$$

But we know $f(7) = 2$, so this doesn't work. I thought it should work but I realized that the steps were taking were not the steps we could take in our question $(1, 2)$. We instead take steps of 2 and 3

III.4 Correct Recurrence Formula

The key insight came from thinking about how we can reach vertex n. Since we can only move forward by 2 or 3, any path ending at n must have come from either $n-2$ or $n-3$.

Therefore:

$$f(n) = f(n-2) + f(n-3)$$

The number of paths to n equals all the paths that reach $n-2$ plus all the paths that reach $n-3$.

Check:

$$\begin{aligned} f(6) &= f(4) + f(3) = 1 + 1 = 2 \quad \checkmark \\ f(7) &= f(5) + f(4) = 1 + 1 = 2 \quad \checkmark \\ f(8) &= f(6) + f(5) = 2 + 1 = 3 \quad \checkmark \end{aligned}$$

This pattern is true for all values I tested!

Now, why does this work:

- Every path ending at n must come from previous vertex
- Since steps are 2 or 3, the only vertices that can reach n must be $n-2$ or $n-3$
- If the number paths to $n-2$ plus the number of paths to $n-3$ will give us the total paths to reach n

This felt odd, because then I thought about what happens if we have an overlap:

III.5 Finding the Sum Formula

The original problem asked for the sum $S(2005) = f(1) + f(2) + \cdots + f(2005)$, not just $f(2005)$. I needed a formula for the sums.

$$
\begin{aligned}
&S(1) = f(1) = 1\\
&S(2) = f(1) + f(2) = 1 + 0 = 1\\
&S(3) = 1 + 0 + 1 = 2\\
&S(4) = 1 + 0 + 1 + 1 = 3\\
&S(5) = 1 + 0 + 1 + 1 + 1 = 4\\
&S(6) = 1 + 0 + 1 + 1 + 1 + 2 = 6
\end{aligned}
$$

Looking at this sequence: $1, 1, 2, 3, 4, 6, 8, 11, 15, \ldots$ My hypothesis was that maybe S(n)= S(n-2) + S(n-3), just like $f(n)$, so checking:

$$
\begin{aligned}
&S(6) \stackrel{?}{=} S(4) + S(3) = 3 + 2 = 5 \quad \text{but } S(6) = 6 \quad \times\\
&S(7) \stackrel{?}{=} S(5) + S(4) = 4 + 3 = 7 \quad \text{but } S(7) = 8 \quad \times\\
&S(8) \stackrel{?}{=} S(6) + S(5) = 6 + 4 = 10 \quad \text{but } S(8) = 11 \quad \times
\end{aligned}
$$

I didn't work, so the sum sequence doesn't follow the same rules as $f(n)$ did. It seemed like since it work for $f(n)$ then it should work with the sums too, but the sums of Fibonacci numbers have their own patterns.

III.6 Deriving the Sum Formula

Let $S(n) = f(1) + f(2) + f(3) + ... + f(n)$. Then $S(n) - S(n-1) = f(n)$.

Since $f(n) = f(n-2) + f(n-3)$, we can get:

$$S(n) - S(n-1) = f(n-2) + f(n-3)$$

I can rewrite the right side using the fact that $f(n) = S(n) - S(n-1)$:

$$
\begin{aligned}
f(n-2) &= S(n-2) - S(n-3)\\
f(n-3) &= S(n-3) - S(n-4)
\end{aligned}
$$

Substituting:

$$
\begin{aligned}
S(n) - S(n-1) &= [S(n-2) - S(n-3)] + [S(n-3) - S(n-4)]\\
S(n) - S(n-1) &= S(n-2) - S(n-4)\\
S(n) &= S(n-1) + S(n-2) - S(n-4)
\end{aligned}
$$

Checking:

$$
\begin{aligned}
&S(5) = S(4) + S(3) - S(1) = 3 + 2 - 1 = 4 \quad \checkmark\\
&S(6) = S(5) + S(4) - S(2) = 4 + 3 - 1 = 6 \quad \checkmark\\
&S(7) = S(6) + S(5) - S(3) = 6 + 4 - 2 = 8 \quad \checkmark
\end{aligned}
$$

The sum of recurrence: $S(n)$ was confusing at first, but it become more understandable when looking how it works:

- $S(n-1)$ gives us the sum of all number before it
- We still need to add $f(n)$ to get $S(n)$ but $f(n) = f(n-2) + f(n-3)$
- When we write these values of f as a difference of S values, it gives us $S(n-2) - S(n-4)$

With this formula in hand, I could compute larger values efficiently. I noticed some interesting things:
The ratio $S(n)/S(n-1)$ stabilizes around 1.33. $S(10)/S(9) = 20/15 \approx 1.33$

I wondered if there might be a simple formula for $S(n)$ in terms of $f(n)$. For instance, the Fibonacci sequence, we have the identity:

$$F(1) + F(2) + \cdots + F(n) = F(n+2) - 1$$

I was wondering could something similar work? So $S(n) = f(n+k) + c$ for some constants k and c. Testing with my computed values from my table above. I notice that $S(9) = 15$ and $f(11) = 7$, so $S(9) \neq f(11) + c$ There doesn't seem to be a simple relationship like $S(n) = f(n+k)$.

This makes sense because our formula looks back at two different vertices ($n-2$ and $n-3$), while Fibonacci only looks back one vertex.

III.7 Computing $S(2005)$

With the recurrence formula, I wrote a simple program to compute $S(2005)$ iteratively. The sequence grows exponentially and the ratio $\frac{S(n)}{S(n-1)}$ stabilizes around 1.33 for large n.

Using the recurrence formula with initial values: $S(1) = 1$, $S(2) = 1$, $S(3) = 2$, $S(4) = 3$, the program computed:

$$S(2005) \approx 9.127 \times 10^{239}$$

IV Conclusions

The answer, $S(2005) \approx 9.127 \times 10^{239}$, to my question was rather disappointing because I originally thought there wouldn't be a big number. However, it turned out to be a really big number and using a computer program to compute it didn't feel right.

This question dived into topics that were unique. Several aspects of this problem surprised me. The investigation required combining graph theory, combinatorics, and algebra that I hadn't thought of when I first read the problem. The question felt very intuitive and simple on the outside, but as you delve into the depth of the question, it became complicated and interesting.

A lot of the aspects of this problem surprised me. First, I didn't expect the recurrence to be $f(n) = f(n-2) + f(n-3)$ rather than the more familiar Fibonacci form $f(n) = f(n-1) + f(n-2)$. My initial thought was wrong because I didn't realized our step distances which reading the question made so much more sense. This really taught me how the way the question is worded can affect the recurrence and instead of jumping to more familiar formulas.

The thing that still surprises me is that $n = 2$ is the only unreachable number and when I first modeled the problem as a graph, I guessed that several small numbers might be unreachable, creating a gap between several numbers. However, the steps turn out to be able to reach all numbers, even the primes.

What I found most entertaining was that how many failed hypothesis and formulas lead me to think harder and do more research. When the Fibonacci formula failed, it wasn't a loss because using that the step distance of the Fibonacci formula was 1 allowed to think about my step sizes and try to apply my own recurrence using that formula.

V Extensions

This problem opens several interesting ideas for further exploration:

V.1 Different Step Sizes

What happens if we allow jumps of different sizes?

Jumps of $\{1, 2\}$: Following the same logic, the recurrence becomes $g(n) = g(n-1) + g(n-2)$, which is the Fibonacci sequence. Every positive integer would be reachable from 1.

Jumps of $\{3, 5\}$: The recurrence would be $h(n) = h(n-3) + h(n-5)$, but many integers would be unreachable. For instance, from 1 we can reach $4, 6, 8 \ldots$ but not $2, 3, 5, 7$.

V.2 General Pattern

For allowed jumps of size a and b where $a < b$, the formula is:

$$f(n) = f(n-a) + f(n-b)$$

An interesting question arises: which numbers are reachable from 1? For our problem with jumps $\{2, 3\}$, we saw that only $n = 2$ is unreachable. With jumps of $\{3, 5\}$, far more numbers would be unreachable (like 2, 3, 4, 7, etc.). The difference is that 2 and 3 are consecutive integers, while 3 and 5 are not, creating more "gaps" in what can be reached.

V.3 Three or More Step Sizes

What if we allow three different jump sizes, say $\{2, 3, 5\}$? The recurrence would become:

$$f(n) = f(n-2) + f(n-3) + f(n-5)$$

This would grow even faster and would be interesting to investigate.

V.4 Negative and Positive Step Sizes

In the original problem, we only move forward. What if we could jump backward too? For instance, from any position n, we could jump to $n \pm 2$ or $n \pm 3$. This would dramatically increase the number of paths and change the problem from finding directed paths to finding more Hamiltonian paths.

VI References

1. The Mandelbrot Competition, October 2005, Round 1, Problem 5
2. "Depth-first search." *Wikipedia*, Wikimedia Foundation, https://en.wikipedia.org/wiki/Depth-first_search
3. "Recurrence relation." *Wikipedia*, Wikimedia Foundation, https://en.wikipedia.org/wiki/Recurrence_relation